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Abstract

We address major computational science issues arising in modeling oil and gas production. We describe a new methodology based on mortar spaces which allow for multiple models and for dynamic upscaling not requiring pseudo functions. Also, we introduce a new Discontinuous Galerkin discretization which is higher-order, locally conservative and allows for non-conforming grids.

1 Introduction

In oil and gas production, the major objective is to maximize return on investment. The challenges involve the ability to treat large detailed flow models, geologic uncertainty, and operational flexibility since infinitely many production strategies are possible. These challenges clearly point to the requirement of accurate and efficient parallel simulators which can be coupled to geostatistical and economic models within a flexible and friendly computational infrastructure.

Geologic uncertainty and production strategies need to be evaluated simultaneously. This involves multiple realizations of multiple geostatistical models and the number and localization of wells as well as the coupling to economic models which are functions of production data, cost/price parameters, rate of return on investment, etc. From a computational point of view, one must be able to treat these uncertainties in a seamless fashion and to test variations on production strategies, evaluate sweep efficiency, and bypassed oil.

At the Center for Subsurface Modeling we have developed a computing framework called IPARS (Integrated Parallel Accurate Reservoir Simulator) for modeling multiphase multiphysics flow in porous media, suitable for massively parallel computers or clusters of PCs. This framework provides all the memory management, message passing, table lookup, solver, and I/O so that a developer only needs to code the relevant physics. This software permits rigorous, physically representative coupling of different physical and numerical flow models in different parts of the domain and accounts for structural discontinuities. In addition, different formulations or physical models can exist on different subdomains. There are currently eight physical models in IPARS as well as a geomechanics capability. This software has also been coupled to a geostatistics package [3] and to various computer science tools: DataCutter for terascale data management/interrogation and DISCOVER for collaborative interactive simulation. The latter couplings have been applied to demonstrate grid computing as well as studying effects of uncertainty on economic models and bypassed oil. Because of page limitations, we will not discuss these results in this paper and the reader is referred to [11].

Here we describe a methodology, mortar space upscaling, and a new family of novel algorithms, Discontinuous Galerkin, for modeling multiphase flow which have been implemented in the IPARS framework. In applying mortar spaces, a reservoir is decomposed into a series of subdomains (blocks) in which independently constructed numerical grids and possibly different physical models and discretization techniques can be employed in each block. Physically meaningful matching conditions are imposed on block interfaces in a numerically stable and accurate way using mortar finite element spaces. Coarse mortar grids and fine subdomain grids provide two-scale approximations. In the resulting effective solution, flow is computed in subdomains on the fine scale while fluxes are matched on the coarse scale. In addition the flexibility to vary adaptively the number of interface degrees of freedom leads to more accurate multiscale approximations. Unlike most upscaling approaches the underlying systems can be treated fully implicitly.

Discontinuous Galerkin (DG) methods involve using discontinuous approximating spaces in which varying degrees of approximation can be defined over each grid block or finite element [10]. The grid itself may be non-conforming, Thus impermeable streaks and channels can be modeled by these high-order methods which are locally conservative. It can be shown that these methods when applied to problems with smooth solutions are nearly exponentially convergent.

The outline of this paper is as follows. In Section 2 we provide a discussion of the multiphase models we are considering. Also, we describe discrete models: mortar space methods for the expanded mixed finite element method for two and three phase flow, and DG for two phase. Numerical results and conclusions are given in Sections 3 and 4, respectively.

2 Multiphase model formulation

In this section we briefly recall the conservation equations and constitutive laws. For brevity, we consider two models - two-phase (water and oil) flow, and three-phase (water, oil, and gas) flow, which commonly occur in reservoir simulation.

We use the following notation. The reservoir is denoted by Ω . The pairing $(\cdot, \cdot)_S$ denotes the scalar product of integrable functions over $S \subseteq \Omega$; $(f, g)_S = \int_S f(x)g(x)dx$. In addition, on lower-dimensional sets like boundary ∂S of a set S we use notation $\langle f, g \rangle_e = \int_e f(s)g(s)ds$ for $e \subseteq \partial S$. We define the jump and average of a function f across e , with $e = \partial E^k \cap \partial E^l$, by $[f] = (f|_{E^k})|_e - (f|_{E^l})|_e$, $\{f\} = \frac{1}{2}(f|_{E^k})|_e + \frac{1}{2}(f|_{E^l})|_e$, respectively.

Let the lower case subscripts $m = w, o$, and $m = g$ denote the water, oil and gas phases respectively and the upper case subscripts $M = W, O$, and $M = G$ the flowing components, water, heavy hydrocarbon or oil, and light hydrocarbon or gas. The corresponding phase saturations are denoted by S_m , viscosities by μ_m , the phase pressures by P_m , and component concentrations by N_M . The porosity ϕ and permeability tensor $\mathbf{K}$ are spatially varying and constant in time reservoir rock data. Other rock properties involve relative permeability k_m and capillary pressure relationships $P_{cm_1m_2}$ which are given functions of saturations and possibly also of position in the case of different *rock types*. We define phase mobilities by $\lambda_m = \frac{k_m}{\mu_m}$. In oil-water models, total mobility is defined as $\lambda_t = \lambda_o + \lambda_w$. The well injection / production rates of the components are denoted q_W, q_O, q_G and are defined using the Peaceman correction extended to multiphase and multicomponent flow.

For each component, we have mass conservation

$$\frac{\partial(\phi N_M)}{\partial t} + \nabla \cdot \mathbf{U}_M = q_M, \quad (1)$$

The system satisfies a volume constraint; namely, that the sum of the saturations equals one. Additional constitutive relations involve the capillary pressure relations, for example, $P_c = P_{cow}(S_w) =$

$P_o - P_w$ and similarly defined P_{ego} . Concentrations and fluxes are defined for each model separately. In this paper, we consider two-phase model and black-oil model.

The two-phase model describes two slightly compressible fluid phases of saturations S_w and S_o , the aqueous and oleic phases respectively, identified with immiscible (pseudo) components water and oil. We have concentrations $N_M = S_m \rho_m = S_m \rho_m^{ref} e^{c_m P_m}$ and fluxes defined by Darcy's law

$$\mathbf{U}_M \equiv \mathbf{U}_m = -\frac{K}{\mu_m} (\rho_m k_m) (\nabla P_m - \rho_m G \nabla D) \quad (2)$$

for each component, $M = O, W$ identified with a phase $m=o, w$, respectively.

The black-oil model is a three-phase (water, oil and gas) model. It is assumed that the aqueous phase contains only water component and that the water component does not exist in other phases. Furthermore, the gas phase contains only the light hydrocarbon component. These are standard assumptions [6, 4]. Note that the gas component flux includes the amount of gas or light hydrocarbon component present in the gas phase as well as the amount dissolved in the oil phase through the gas-oil ratio $R_o = \frac{N_G}{N_O}$ which is bounded from above by pressure dependent solution gas-oil ratio at saturated conditions R_{so} given as reservoir data. Specifically, if the pressure in the reservoir is high enough (above the *bubble point*) then all of the light hydrocarbon component is dissolved in the (undersaturated) oil phase with $N_G \leq R_{so} N_O$. Otherwise, $N_G > R_{so} N_O$: and the reservoir is in three-phase conditions with oil phase saturated with gas. We omit the equations for lack of space and refer to [4, 5] for details.

Below we consider in detail two fully discrete models: two-phase implicit formulation and its mortar multiblock form, and a sequential formulation, and its fully discrete DG counterpart.

2.1 Two-phase implicit model

The discrete-in-time implicit mass conservation equation and Darcy's law (1)-(2) read

$$\frac{(\phi N_M)^{n+1} - (\phi N_M)^n}{\Delta t_{n+1}} + \nabla \cdot \mathbf{U}_M^{n+1} = q_M^{n+1}, \quad (3)$$

The system is obtained by the backward Euler formula and is solved for P_o, N_O : this choice gives all the other variables directly from functional relationships including capillary pressure. In the implicit black-oil model (equations not given for lack of space) considered here, the primary variables are water pressure P_w , oil component concentration N_O , and gas component concentration N_G . In this case, all the other variables have to be obtained from primary unknowns by iteration.

We assume for simplicity that no flow boundary condition is imposed on $\partial\Omega$, although more general types of boundary conditions can also be treated. Also, we decompose the simulation domain $\Omega \subset \mathcal{R}^3$ into a series of subdomains Ω_k , $k = 1, \dots, n_b$. Let $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ be the interface between Ω_k and Ω_l . A physical model and separate discretization is associated with each block. In addition, a condition on each interface is formulated. On each Γ_{kl} the following physically meaningful continuity conditions are imposed: $P_m|_{\Omega_k} = P_m|_{\Omega_l}$ on Γ_{kl} , $[\mathbf{U}_M \cdot \nu]_{kl} \equiv \mathbf{U}_M|_{\Omega_k} \cdot \nu_k + \mathbf{U}_M|_{\Omega_l} \cdot \nu_l = 0$ on Γ_{kl} . where ν_k denotes the outward unit normal vector on $\partial\Omega_k$. These equations represent continuity of pressures and normal fluxes, respectively. The sets of phase indexes m and component indexes M for which the above conditions are imposed depend on the physical models imposed on the neighboring blocks. The above conditions lead to well posed systems only when all relevant phases are present. Otherwise, some other interface conditions involving variables other than pressures must be imposed. In addition, when these conditions are discretized, the matching occurs in an average (weak) sense, see below.

The space discretization is based on mixed finite element methods reduced to cell-centered finite differences, see below. The nonlinear or quasilinear system of equations arising in some models are solved by the Newton method and the Jacobian equation is solved by one of the suite of iterative solvers capable of handling non-symmetric and non-positive systems arising from the Jacobian, for example preconditioned GMRES. The Newton method stops when the residuals are less than a prescribed tolerance.

A mortar mixed finite element method for two-phase implicit two-phase model. The domain decomposition formulation for mixed methods described here was extended to the non-matching grids with mortar spaces in [1]; multinumerics and multiphysics applications can be found in [4, 11] and in refs. therein.

The discretization in space on each Ω_k is achieved through the use of the lowest order Raviart-Thomas mixed finite element spaces RT_0 [8] on a rectangular partition of Ω_k by $\mathcal{T}_{h_k}$, where h_k is associated with the size of the elements. In fact, we employ the *expanded mixed method* which has been developed for accurate and efficient treatment of irregular domains [2, 7]. We use spaces V_{h_k}, W_{h_k} for velocities and scalar variables, respectively, see [8] for details. In addition, we use the mortar finite element space $M_{h_{kl}}$ defined on a rectangular grid $\mathcal{T}_{h_{kl}}$ on Γ_{kl} , where h_{kl} is associated with the size of the elements in $\mathcal{T}_{h_{kl}}$. If the grids adjacent to Γ_{kl} are non-matching, the interface grid need not match either of them. The mortar space is made of tensor product of finite element piecewise linears defined in each space dimension. In this space we approximate the interface pressures and concentrations, and impose weakly normal continuity of fluxes.

For $m = w, o$, we introduce $\tilde{U}_m$

$$\tilde{U}_m = -\nabla P_m, \quad U_m = \frac{k_m K}{\mu_m} \rho_m (\tilde{U}_m + \rho_m G \nabla D).$$

The discretized equation (3) on a subdomain Ω_k is solved for $U_{h,m}^{n+1}|_{\Omega_k} \in V_{h_k}$, $\tilde{U}_{h,m}^{n+1}|_{\Omega_k} \in V_{h_k}$, $P_{h,m}^{n+1}|_{\Omega_k} \in W_{h_k}$, $N_{h,m}^{n+1}|_{\Omega_k} \in W_{h_k}$, and $P_{h,m}^{M,n+1}|_{\Gamma_{kl}} \in M_{h_{kl}}$ which satisfy, for $1 \leq k < l \leq n_b$, $M = O, W$, and $m = o, w$,

$$\left(\frac{(\phi N_{h,M})^{n+1} - (\phi N_{h,M})^n}{\Delta t^{n+1}}, w \right)_{\Omega_k} + (\nabla \cdot U_{h,m}^{n+1}, w)_{\Omega_k} = (q_m, w)_{\Omega_k} \quad w \in W_{h_k}, \quad (4)$$

$$(\tilde{U}_{h,m}^{n+1}, v)_{\Omega_k} = (P_{h,m}^{n+1}, \nabla \cdot v)_{\Omega_k} - \left\langle P_{h,m}^{M,n+1}, v \cdot \nu_k \right\rangle_{\partial\Omega_k \setminus \partial\Omega} \quad v \in V_{h_k}, \quad (5)$$

$$(U_{h,m}^{n+1}, \tilde{v})_{\Omega_k} = \left(\frac{k_{h,m}^{n+1} K}{\mu_{h,m}} \rho_{h,m}^{n+1} (\tilde{U}_{h,m}^{n+1} + \rho_{h,m}^{n+1} G \nabla D), \tilde{v} \right)_{\Omega_k} \quad \tilde{v} \in V_{h_k}, \quad (6)$$

$$\left\langle [U_{h,m}^{n+1} \cdot \nu]_{kl}, \mu \right\rangle_{\Gamma_{kl}} = 0, \quad \mu \in M_{h_{kl}}. \quad (7)$$

The last equation describes the weak (average) matching of fluxes.

Introducing the pressure gradients $\tilde{U}_m$ in the expanded mixed method allows for proper handling of the degenerate (for $S_m = 0$) relative permeability $k_m(S_m)$ in (5)–(6) (the edge values are computed by upwinding). It also allows, even for a full permeability tensor K , to accurately approximate the mixed method on each subdomain by cell-centered finite differences for P_o and N_o . This is achieved by approximating the vector integrals in (5) and (6) by a trapezoidal quadrature rule and eliminating $\tilde{U}_{h,m}$ and $U_{h,m}$ from the system [2].

Reduction to an interface problem. For simplicity consider two blocks A and B with interface I. In the multiblock multimodel approach we can assign one model to the domain (block) A and possibly a different model to the block B. We seek the interface values of the *primary*

unknowns or *primary variables* (denoted by Λ) on I such that the fluxes of components match in a prescribed weak sense.

In other words, we solve the problem $B(\Lambda) = 0$ (equivalent to (7)) where $B(\Lambda)$ denotes the jump across I (or its average with respect to the mortar space) of component fluxes. The choice of the primary unknowns on the interface is problem dependent and it is motivated by the convergence properties, computational efficiency, or coding convenience, see [4, 7] and refs. therein. In the case of (7) we use (P_o, N_o) as the interface primary variables. At every time step t_{n+1} , we seek the interface values $\Lambda = \Lambda^{n+1} = (P_o^{*,n+1}, N_o^{*,n+1})$ so that $|B(\Lambda)| < \xi$, where ξ is some prescribed tolerance and $|\cdot|$ is a suitable norm. The interface problem solution is found by using a Newton–Krylov iteration in which the Jacobian $B'(\Lambda)$ is approximated by a finite difference. The evaluation of $B(\Lambda)$ requires the solution in parallel of subdomain problems with Dirichlet data Λ , in other words, solving (4–6). Once the subdomain problem is solved, the normal fluxes of oil and water and $B(\Lambda)$ are computed. See [7] for details and refs.

2.2 Two-phase sequential model

The IPARS two-phase sequential model relies on the splitting of the model equations into the time lagged (formally) elliptic part and the parabolic–hyperbolic part. Other formulations related to the one proposed here are those of *total pressure* or *streamlines–streamtubes* methods; several others exist. The main idea behind the formulation is that even though the individual phase mobilities vary strongly with the water content function, the sum of them (the total mobility) remains close to a constant over many time steps. Therefore the pressure profiles remain stable even though water saturation varies strongly. Once the pressure profiles are known, one solves for the saturation analytically (along streamlines) or numerically as we show below.

Consider for simplicity the incompressible case and omit the well terms. The primary variables in the sequential formulation are water pressure P_w and saturation S_w . The first equation defines the value of water pressure at the new time step P_w^{n+1} as a solution to the problem

$$-\nabla \cdot (K \lambda_t^n (\nabla P_w^{n+1})) = \nabla \cdot (K \lambda_o^n \nabla P_c(S_w^n)) - \nabla \cdot (K (\lambda_o^n \rho_o^n + \lambda_w^n \rho_w^n) G \nabla D). \quad (8)$$

This equation arises after the mass conservation equations for two phases are added and discretized in time. Using the values of P_w^{n+1} the densities, the phase velocities U_o, U_w and the total velocity $U_t = U_o + U_w$ are computed. Then the saturation equation is solved for S_w^{n+1}

$$\begin{aligned} \frac{\phi S_w^{n+1}}{\Delta t_{n+1}^S} + \nabla \cdot [K \frac{\lambda_o^n \lambda_w^n}{\lambda_t^n} P'_c(S_w^n) \nabla S_w^{n+1}] &= \frac{\phi}{\Delta t_{n+1}^S} S_w^n \\ - \nabla \cdot [\frac{\lambda_w^n}{\lambda_t^n} U_t^n] - \nabla \cdot [K \frac{\lambda_o^n \lambda_w^n}{\lambda_t^n} (\rho_w^n - \rho_o^n) G \nabla D] &= 0. \end{aligned} \quad (9)$$

The time step for saturation may be much smaller than the pressure step or, alternatively, pressure solution can be skipped and redone only every M saturation steps, with M as large as 100. The saturation time step is limited by the CFL–type stability condition.

This model can be discretized in space similarly as the implicit model shown above using mixed or mixed multiblock formulation. Or, we can use DG formulation.

Discontinuous Galerkin discretization. Here we define the discontinuous Galerkin scheme for solving (8)–(9). Let $\mathcal{E}_h = \{E\}_E$ be a non degenerate subdivision of Ω , made of triangles or quadrilaterals in 2D and tetrahedra or prisms in 3D. Let h be the maximum diameter of the elements. Let Γ_h be the union of the open sets that coincide with interior edges (or interior faces)

of elements of $\mathcal{E}_h$. Let e denote a segment of Γ_h shared by two triangles E^k and E^l of $\mathcal{E}_h$; we associate with e , once and for all, a unit normal vector $\mathbf{n}_e$ directed from E^k to E^l .

For each integer r , we associate a finite element subspace of discontinuous piecewise polynomials: $\mathcal{D}_r = \{v : v|_E \in P_r(E) \quad \forall E \in \mathcal{E}_h\}$, where $P_r(E)$ is a discrete space containing the set of polynomials of total degree less than or equal to r on E . We will approximate the wetting phase pressure and saturation by discontinuous polynomials of different orders. Let r_p and r_s be the degrees of polynomials for the pressure and saturation respectively.

Based on (8) and (9), we formulate the following discrete scheme: given $(P_w^n, S_w^n) \in \mathcal{D}_{r_p} \times \mathcal{D}_{r_s}$, find $(P_w^{n+1}, S_w^{n+1}) \in \mathcal{D}_{r_p} \times \mathcal{D}_{r_s}$ such that for all $(v, w) \in \mathcal{D}_{r_p} \times \mathcal{D}_{r_s}$:

$$\begin{aligned} & \sum_{E \in \mathcal{E}_h} (K \lambda_t^n \nabla P_w^{n+1}, \nabla v)_E - \sum_{e \in \Gamma_h \cup \Gamma_D} \langle \{K \lambda_t^n \nabla P_w^{n+1} \cdot \mathbf{n}_e\}, [v] \rangle_e \\ & + \sum_{e \in \Gamma_h \cup \Gamma_D} \langle \{K \lambda_t^n \nabla v \cdot \mathbf{n}_e\}, [P_w^{n+1}] \rangle_e = \sum_{E \in \mathcal{E}_h} (\chi_h^n, \nabla v)_E \\ & - \sum_{e \in \Gamma_h \cup \partial \Omega} \langle \tilde{\chi}_h^n \cdot \mathbf{n}_e, [v] \rangle_e + \sum_{e \in \Gamma_D} \langle (K \lambda_t^n \nabla v \cdot \mathbf{n}), p_{\text{dir}} \rangle_e \\ & (\frac{\phi}{\Delta t^{n+1}} S_w^{n+1}, z) + \sum_{E \in \mathcal{E}_h} (K \frac{\lambda_o^n \lambda_w^n}{\lambda_t^n} |P'_c(S_w^n)| \nabla S_w^{n+1}, \nabla z)_E - \sum_{e \in \Gamma_{\text{in}}} \langle S_w^{n+1} U_t^i \cdot \mathbf{n}_e, z \rangle_e \\ & - \sum_{e \in \Gamma_h} \langle \{K \frac{\lambda_o^n \lambda_w^n}{\lambda_t^n} |P'_c(S_w^n)| \nabla S_w^{n+1} \cdot \mathbf{n}_e\}, [z] \rangle_e + \sum_{e \in \Gamma_h} \langle \{K \frac{\lambda_o^n \lambda_w^n}{\lambda_t^n} |P'_c(S_w^n)| \nabla z \cdot \mathbf{n}_e\}, [S_w^{n+1}] \rangle_e \\ & = (\frac{\phi}{\Delta t^{n+1}} S_w^n, z) + \sum_{E \in \mathcal{E}_h} (\zeta_h^n, \nabla z)_E - \sum_{e \in \Gamma_h \cup \Gamma_D} \langle \tilde{\zeta}_h^n \cdot \mathbf{n}_e, [z] \rangle_e \\ & - \sum_{e \in \Gamma_h \cup \Gamma_D} \langle \frac{\lambda_w^n}{\lambda_t^n} \{K \lambda_t^n \nabla z \cdot \mathbf{n}_e\}, [P_w^n] \rangle_e - \sum_{e \in \Gamma_{\text{in}}} \langle s_{\text{in}} U_t^n \cdot \mathbf{n}_e, z \rangle_e, \end{aligned}$$

where U_t^i , ζ_h^i and χ_h^i are the approximate quantities defined below

$$\begin{aligned} U_t^n &= -K \lambda_w^n (\nabla P_w^n - \rho_w G \nabla D) - K \lambda_o^n (P'_c(S_w^n) \nabla S_w^n + \nabla P_w^n - \rho_o G \nabla D) \\ \chi_h^n &= K \lambda_o^n |P'_c(S_w^n)| \nabla S_w^n + K (\lambda_o^n \rho_o + \lambda_w^n \rho_w) G \nabla D \\ \zeta_h^n &= \frac{\lambda_w^n}{\lambda_t^n} \{U_t^n\} + K \frac{\lambda_o^n \lambda_w^n}{\lambda_t^n} (\rho_w - \rho_o) G \nabla D. \end{aligned}$$

The quantities $\tilde{\chi}_h$ and $\tilde{\zeta}_h$ are upwind values of χ_h and ζ_h with respect to $\{U_t\} \cdot \mathbf{n}$. By fixing an element E and a test function $v \in \mathcal{D}_{r_p}$ that vanishes outside of E , we easily see that the method is locally conservative.

3 Numerical examples.

DG for two phase flow. On Fig. 1, we show the saturation and pressure contours corresponding to a quarter-five spot problem. The mesh consists of triangular elements. The uniform permeability tensor $\mathbf{K}$ is defined by $K_{xx} = 400$, $K_{yy} = 100$ and $K_{xy} = K_{yx} = 50$.

Multiblock mixed coupling of two-phase and black-oil models. The permeability field (2D view), pressure and well locations are shown in Fig. 2. Initial gas concentration and multiblock

Figure 1: Quarter-five spot problem with full permeability tensor. Linear saturation and quadratic pressure at 100 days

decomposition as well as gas distribution after 3 years of production are also shown there. In this example we use black-oil model in the center block and two-phase model in all the remaining 8 blocks.

4 Conclusions

We have described a methodology called mortar upscaling for modeling and coupling complex multiphysics phenomena which allows for treating different physics and different time discretizations in different subdomains. Since the number of interface degrees of freedom on the interface can be chosen adaptively this method can be viewed as a dynamic subgrid-scale modeling approach. In addition, we defined a family of schemes which are based on discontinuous approximating spaces. The Discontinuous Galerkin schemes are higher-order accurate and are locally conservative. It has been shown in [9] that these algorithms can be coupled with cell-centered finite differences using either mortar or mortar-free methods.

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Figure 2: IPARS multiblock example

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